The Dirichlet Function

Bailey Whitbread

University of Queensland

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Definition

Named after 19th century German mathematician Johann Peter Gustav Lejeune Dirichlet.

\[ D : [0, 1] \rightarrow \mathbb{R}, \quad D(x) = \begin{cases} 
1, & \text{if } x \in \mathbb{Q} \\
0, & \text{else}
\end{cases} \]

i.e. if \( x = \frac{p}{q} \) for some \( p, q \in \mathbb{Z} \) then \( D(x) = 1 \), else \( D(x) = 0 \).

e.g. \( D(\frac{1}{2}) = 1, \ D(0.34) = 1, \ D(\frac{1}{\sqrt{5}}) = 0, \ D(\frac{\pi}{6}) = 0. \)
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\[ D(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{Q} \\
0 & \text{else}
\end{cases} \]
Definition

A partition \( P \) of \([a, b] = \{x : a \leq x \leq b\}\) is a finite sequence \((x_0, x_1, ..., x_{n-1}, x_n)\) where

\[
a = x_0 < x_1 < ... < x_{n-1} < x_n = b
\]

This cuts up \([a, b]\) into \(n\) blocks.

e.g. One partition of \([0, 5]\) is \(P = (0, 1, 3.5, 5)\) where \(n = 3\) and \(x_0 = 0, x_1 = 1, x_2 = 3.5\) and \(x_3 = 5\).

Denote the \(i^{th}\) block by \(B_i = [x_{i-1}, x_i]\) which has width \(W_i = x_i - x_{i-1}\).
Riemann Integration

Definition

Consider a bounded function $f : [a, b] \to \mathbb{R}$ and a partition $P$ of $[a, b]$. The upper sum of $f$ with respect to $P$ is the number

$$U_{f,P} = \sum_{i=1}^{n} W_i \sup_{x \in B_i} f(x)$$

Similarly, the lower sum of $f$ with respect to $P$ is the number

$$L_{f,P} = \sum_{i=1}^{n} W_i \inf_{x \in B_i} f(x)$$
Riemann Integration

Figure: Upper and lower sums.
Riemann Integration

Definition

We say $f$ is *Riemann integrable* if, for all $\varepsilon > 0$, there exists a partition $P$ such that

$$U_{f,P} - L_{f,P} < \varepsilon$$
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Is It Riemann Integrable?

For any partition $P$ of $[0, 1]$, consider any block $B_i$ of that partition.

We see that $D$ takes values 0 and 1, no matter the block’s width.

This is because the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{I}$ are ‘dense’ in the real numbers $\mathbb{R}$.

And so, $U_{D,P} = 1$ and $L_{D,P} = 0$ for any partition $P$, so then

$$U_{D,P} - L_{D,P} = 1 \nless \varepsilon$$

and $D$ is not Riemann integrable.
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Closed Form

\[ D(x) = \lim_{k \to \infty} \left( \lim_{j \to \infty} \cos^{2j}(k! \pi x) \right) \]

Proof

Consider the function \( f(x) = \cos(\pi x) \). For integers \( k \), \( f(k) = \pm 1 \).
Proof

Take $g(x) = f(x)^2 = \cos^2(\pi x)$ so that $g(k) = 1$ for integers $k$.

For non-integers $q$, $|f(q)| = |\cos(\pi q)| < 1$, so $0 \leq g(q) < 1$.

If we take the square many times, $g(q) \to 0$ for non-integers $q$, while $g(k) \to 1$ for integers $k$. [Desmos.]
Proof

Then define

\[ Z(x) = \lim_{j \to \infty} g(x)^j = \lim_{j \to \infty} \cos^{2j}(\pi x) = \begin{cases} 1, & \text{if } x \in \mathbb{Z} \\ 0, & \text{else} \end{cases} \]

Remembering our proposed expression for \( D \), we now have

\[ D(x) = \lim_{k \to \infty} \left( \lim_{j \to \infty} \cos^{2j}(k! \pi x) \right) = \lim_{k \to \infty} Z(k!x) \]

Left To Show

\[ D(x) = \lim_{k \to \infty} Z(k!x) \]
Proof

Take $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. Take any integer $k > q$. Now consider

$$Z(k!x) = Z(k! \cdot \frac{p}{q}) = Z\left((1 \cdot 2 \cdot \ldots \cdot q \cdot \ldots \cdot k) \cdot \frac{p}{q}\right)$$

$$= Z\left((1 \cdot 2 \cdot \ldots \cdot 1 \cdot \ldots \cdot k) \cdot p\right)$$

$$= 1$$

This means that $Z(k!x) = 1$ for all $k$ greater than $q$. Then we know $D(x) = \lim_{k \to \infty} Z(k!x)$ for $x \in \mathbb{Q}$!
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**Proof**

Take $x \notin \mathbb{Q}$. Then $k!x \notin \mathbb{Q}$ for all $k$.

So there must hold $Z(k!x) = 0$ for all $k$.

Then we know $\lim_{k \to \infty} Z(k!x) = 0$ for $x \notin \mathbb{Q}$!

So we’ve shown

$$D(x) = \lim_{j,k \to \infty} \cos^{2j}(k!\pi x)$$

as desired. □
What Have We Done?

- $D$ is not Riemann integrable.
- But $D(x) = \lim_{j,k \to \infty} \cos^{2j}(k!\pi x)$
- $\cos^{2j}(k!\pi x)$ is Riemann integrable for each $j, k = 0, 1, 2, \ldots$
- Riemann integrability is not closed under limits!
The Monotone Convergence Theorem

Theorem [MCT for Sequences]
Let \((a_n)_{n=1}^{\infty}\) be a monotone and bounded sequence in \(\mathbb{R}\). Then it converges to an element of \(\mathbb{R}\).

Theorem [MCT for Functions]
Let \((f_n)_{n=1}^{\infty}\) be a monotone and bounded sequence in some function space \(S\). Then it converges to an element of \(S\).
The Monotone Convergence Theorem

The Dirichlet Function

Let \((a_n)_{n=1}^{\infty}\) be an enumeration of \(\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} : x \geq 0\}\).

Then define the sequence of functions

\[
f_n(x) = \begin{cases} 
1, & \text{if } x = a_j \text{ for some } j \leq n \\
0, & \text{else}
\end{cases}
\]

Each \(f_n\) is Riemann integrable since they each have only finitely many nonzero values.
The Monotone Convergence Theorem

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\[ f_n(x) = \begin{cases} 
1, & \text{if } x = a_j \text{ for some } j \leq n \\
0, & \text{else} 
\end{cases} \]

Observe that

\[ 0 \leq f_n(x) \leq f_{n+1}(x) \leq 1 \]

for all \( n \) and for all \( x \). So we have a monotone and bounded sequence. Also observe that

\[ \lim_{n \to \infty} f_n(x) = D(x) \]